Calculation techniques in combining uncertainties

From the face expressions of participants attending my measurement uncertainty courses, I could tell that some of them had yet to grasp the important point of calculating the combined uncertainty from a series of uncertainty components. I hope the following notes can bring more clarity to this issue.

When we are presented with few standard uncertainties $u(x_i)$, expressed in standard deviations from an uncertainty contributing component, we need to find out what the combined standard uncertainty of this component is. The best approach to begin with is to remember a basic definition that "the squared standard deviation is a variance".

So, the combined standard uncertainty can first be evaluated by its total or combined variance which is, of course, the sum of various variances from the uncertainty component. This is referred to as the Law of Propagation of Uncertainty.

Mathematically, we can express it as in equation [1], assuming that each uncertainty contribution is independent:

$$u(y)^2 = \sum c_i{}^2 u(x_i)^2 \qquad [1]$$

where $c_i = \left(\frac{\partial y}{\partial x_i}\right)$ is the sensitive coefficient of x_i.

This equation leads us to two simple situations:

- If a quantity x_i is simply added to or subtracted from all the others to obtain the result y, the contribution to the uncertainty in y is simply the uncertainty $u(x_i)$ in x_i. For example,

  given $y = x_1 + x_2$ with uncertainties $u(x_1)$ and $u(x_2)$, then

$$u(y)^2 = u(x_1)^2 + u(x_2)^2 \text{ , or, } \quad u(y) = \sqrt{u(x_1)^2 + u(x_2)^2} \qquad [2]$$

- If a quantity is multiplied by, or divides, the rest of the expression for y, the contribution to the *relative* uncertainty in y, $u(y)/y$, is the *relative* uncertainty $u(x_i)/x_i$ in x_i. For example,

given $y = \frac{x_1}{x_2}$, then

$$\left(\frac{u(y)}{y}\right)^2 = \left(\frac{u(x_1)}{x_1}\right)^2 + \left(\frac{u(x_2)}{x_2}\right)^2 \quad \text{or}$$

$$u(y) = y \times \sqrt{\left(\frac{u(x_1)}{x_1}\right)^2 + \left(\frac{u(x_2)}{x_2}\right)^2} \qquad [3]$$

The question now is how do we derive the equation [3] from the equation [1]. This can be answered by looking at a simple calculation model example as shown below.

We know the density of a substance is $\rho = \frac{m}{V}$ [4]

where m is the mass of substance and V, its volume. Therefore, the combined uncertainty is

$$u(\rho)^2 = \left(\frac{\partial\rho}{\partial m}\right)^2 (u_m)^2 + \left(\frac{\partial\rho}{\partial V}\right)^2 (u_V)^2 \qquad [5]$$

Now, by differentiating the equation [4], we have: $\frac{d\rho}{dm} = \frac{1}{V}$ and $\frac{d\rho}{dV} = \frac{-m}{V^2}$

So, equation [5] becomes $u(\rho)^2 = \left(\frac{1}{V}\right)^2 (u_m)^2 + \left(\frac{-m}{V^2}\right)^2 (u_V)^2$ [5a]

When equation [5a] is divided by equation [4] on both sides of the equation, we should get: $\left(\frac{u(\rho)}{\rho}\right)^2 = \left(\frac{u_m}{m}\right)^2 + \left(\frac{u_V}{V}\right)^2$ or,

$$u(\rho) = \rho \times \sqrt{\left(\frac{u_m}{m}\right)^2 + \left(\frac{u_V}{V}\right)^2}$$

It is therefore correct that for a case of multiplication and division, we have to work on the relative standard uncertainty or coefficient of variation, CV.