Understanding the Binomial Probability Distribution (Part I)

The binomial probability distribution is a good example of the discrete probability distribution. It is associated with a multiple-step experiment that is called the binomial experiment, which has the following properties:

- The experiment consists of a sequence of n identical trials
- Only two outcomes are possible on each trial: e.g. *success* and *failure, positive* and *negative*
- The probability of an outcome, say success, denoted by p, does not change from trial to trial. Consequently, the probability of another outcome, say failure, denoted by $(1-p)$, does not change from trial to trial as well
- All the trials carried out are independent

Note: If only the last three properties listed above are present, we say the trails are generated by a Bernoulli process. If all the above properties are valid, we say we have a binomial experiment.

A binomial experiment can be easily demonstrated by flipping a coil n times, of which the process is independent. The outcomes can only be either a Head or a Tail. The probability p of any of the two outcomes by flipping a fair coin is 0.5. In this case, the random variable is x = the number of Heads appearing in the n trials, that is x can assume values of 0, 1, 2, …, n.

Suppose we were to carry out a binomial experiment of coin flipping for 10 times and obtain 8 Heads. Is the coin biased? In fact, the Binomial model can answer by giving us the evaluation, given that $p = 0.5$, of obtaining more than 8 Heads or fewer than 2 Heads in 10 trials. But *how do we do this*?

Let's revise our knowledge of basic algebra first. The number of experimental outcomes resulting in exactly x successes in n trials can be computed using the following combination formula:

$$^{n}C_{x} = \frac{n!}{x!(n-x)!}$$

where $n! = n(n-1)(n-2)\ldots(3)(2)(1)$

and, by definition $0! = 1$

In the above example, $n = 10$, $x = 8$ Heads, the Binomial probability function is

$$f(x) = {}^{n}C_{x} p^{x} (1-p)^{(n-x)}$$

where $f(x)$ = the probability of x successes in n trials

n = number of trials

$$^{n}C_{x} = \frac{n!}{x!(n-x)!}$$

p = the probability of a success on any one trial

$(1-p)$ = the probability of a failure on any one trial

Hence, by calculation, we obtain :

$$f(x) = \frac{10!}{8!(10-8)!}(0.5)^{8}(1-0.5)^{2} = 0.044$$

The result indicates that the probability of having 8 heads out of 10 flips is 0.044 or 4.4%.

The table below summarizes a Binomial probability distribution for the coin flipping experiment with n = 10, x = 0 to 10 Heads (H) by using the above logic:

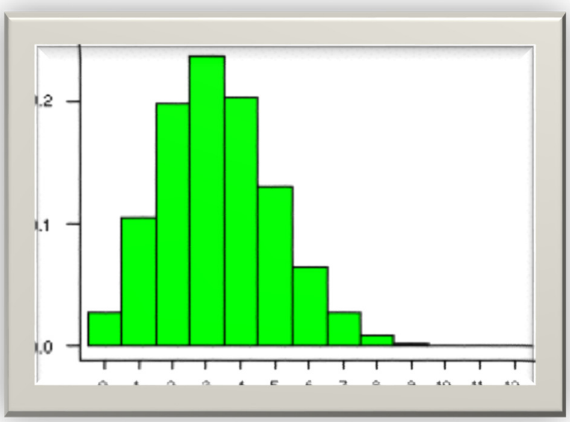
H	T	*f(x)*
0	10	0.0010
1	9	0.0098
2	8	0.0439
3	7	0.1172
4	6	0.2051
5	5	0.2461
6	4	0.2051
7	3	0.1172
8	2	0.0439
9	1	0.0098
10	0	0.0010

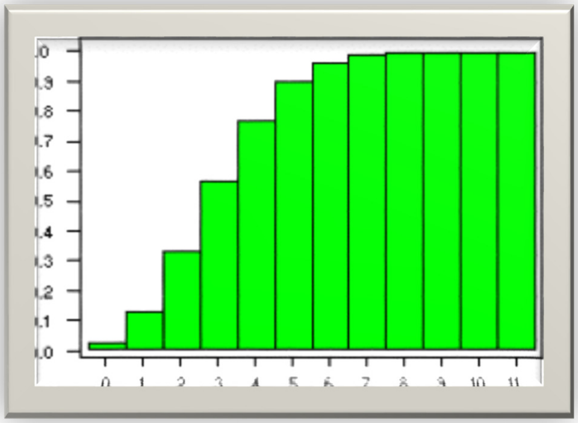
Similarly if we have to roll a perfect single 6-faced die for 20 times (n=20) with a probability p of 1/6 or 1.66667, we have a *B(n=20,p=1/6)* distribution and would obtain the following cumulative binomial distribution:

n	*P(X≤n)*
0	0.0261
1	0.1304
2	0.3287
3	0.5665
4	0.7687
5	0.8982
6	0.9629
7	0.9887
8	0.9972
9	0.9994
10	0.9999

Note that a cumulative binomial probability refers to the probability that the binomial random variable falls within a specified range, i.e. either greater or equal to a stated lower limit and less than or equal to a stated upper limit.

The corresponding graphs for the probability density function and cumulative distribution function for the *B(20,1/6)* distribution are shown below:





So, if we want to know the probability of rolling more than 2 ones in 20 rolls, the probability $P(X>2)$ is equal to $1-P(X<2)$ or equal to:

$$1- P(X=0) + P(X=1) + P(X=2).$$

From the above table, we see that since the probability of 2 or fewer ones is equal to 0.3287, the probability of rolling more than 2 ones = 1–0.3287 = 0.6713.   In other words, we have more than 99% chance to roll out 2 ones when we were to throw the single die more than 7 times.