Design of Experiment –
How to transform experimental data with unequal variances?

In our discussion on experiments whose data are described or analyzed by the one-way analysis of variance ANOVA model, one of the assumption questions we asked was "*do the error variables ε_{it} have similar variances for each treatment?*".

If they do, we have satisfied one of the requirements that the experimental data are of constant variances statistically. If they do not, we should then try to find a variance stabilized transformation of the data to equalize the variances of the error variables. This involves finding some function $h(y_{it})$ of the data so that the model

$$h(Y_{it}) = \mu^* + \tau_i^* + \varepsilon_{it}^*$$

holds and $\varepsilon_{it}^* \sim N(0,\sigma^2)$ and the ε_{it}^*'s are mutually independent for all $t = 1,2,\ldots,r_i$ and $i = 1,2,\ldots, n$.

An approximate transformation can generally be found if there is a clear relationship between the error variance $\sigma_i^2 = \mathrm{Var}(\varepsilon_{it})$ and the mean response $\mathrm{E}[Y_{it}] = \mu + \tau_i$ for $i = 1,\ldots, n$.

If the variance and the mean increase together, as suggested by the megaphone-shaped residual plot against the mean values, or if one increases as the other decreases, then the relationship between σ_i^2 and $\mu + \tau_i$ is often of the form

$$\sigma_i^2 = k(\mu + \tau_i)^q \qquad (1)$$

where k and q are constants. In this case, the function $h(y_{it})$ should be chosen to be

$$h(y_{it}) = \begin{cases} (y_{it})^{1-q/2} & \text{if } q \neq 2, \\ \ln(y_{it}) & \text{if } q = 2 \text{ and all } y_{it}\text{'s are non-zero,} \\ \ln(y_{it}+1) & \text{if } q = 2 \text{ and some } y_{it}\text{'s are zero.} \end{cases} \qquad (2)$$

Usually the value of q is not known but we can find a reasonable approximation empirically as described below.

First we substitute the least square estimates for the parameters into equation (1) and take logs of both sides, giving

$$\ln(s_i^2) = \ln(k) + q(\ln(\overline{y}_i))$$

Therefore, the slope of the line obtained by plotting $\ln(s_i^2)$ against $\ln(\overline{y}_i)$ gives an estimate of q. We shall illustrate this in the following example.

But, it is to be noted that when a transformation is found that equalizes the variances, it is necessary to check or recheck the other model assumptions, because a transformation that cures one problem could cause others. If there are no problem with the other assumptions, we can then proceed with the data analysis as usual.

Example: Battery lifetime experiment

An experiment was designed to find out which type of non-rechargeable battery was the most economical in terms of its lifetime per unit cost. There were two treatment factors each having two levels, namely battery "duty" (level 1=alkaline, level 2 = heavy duty) and "brand" (level 1 = A brand and level 2 = B brand). Hence, the codes for the experiments were:

Code	Treatment combination
1	alkaline, A brand
2	alkaline, B brand
3	heavy duty, A brand
4	heavy duty, B brand

The data for the battery lifetime experiment were tabulated in Table 1 below.

Table 1:
Data for the battery lifetime experiment (hrs)

Code	1	2	3	4
	602	863	232	235
	529	743	255	282
	534	773	200	238
	585	840	215	228
Mean	562.5	804.75	225.5	245.75
Std Dev	36.52	56.14	23.61	24.53
Variance	1333.67	3152.25	557.67	601.58

In this one-way analysis of variance model, the residual (error variable) is taken as $\varepsilon_{it} = y_{it} - \overline{y}_i$ and the standardization of these residuals is achieved by dividing the residuals by their standard error, that is by $\sqrt{\frac{SSE}{n-1}}$. Table 2 summarizes the residuals and standardized residuals for the battery lifetime experiment.

Table 2: Residuals & standardized residuals for the battery experiment

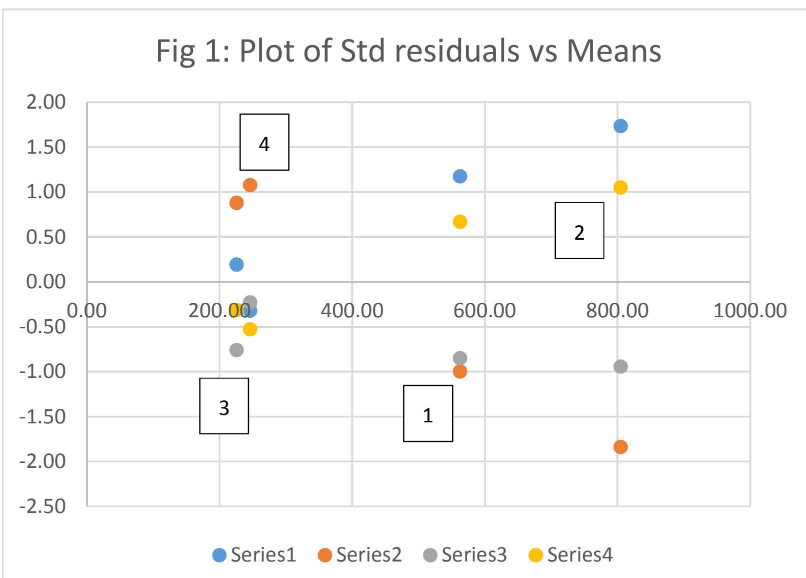
Residuals ε_{it}

Treatment	1	2	3	4
	39.5	58.25	6.5	-10.75
	-33.5	-61.75	29.5	36.25
	-28.5	-31.75	-25.5	-7.75
	22.5	35.25	-10.5	-17.75

Standardized residuals z_{it}

Treatment	1	2	3	4
	1.18	1.73	0.19	-0.32
	-1.00	-1.84	0.88	1.08
	-0.85	-0.94	-0.76	-0.23
	0.67	1.05	-0.31	-0.53

Figure 1 below shows a residual plot of the battery experiment against the four treatments.



The Figure 1 indicates the most common pattern of non-constant variance in which the error variance increases as the mean response increases, with its plot showing a megaphone in shape.

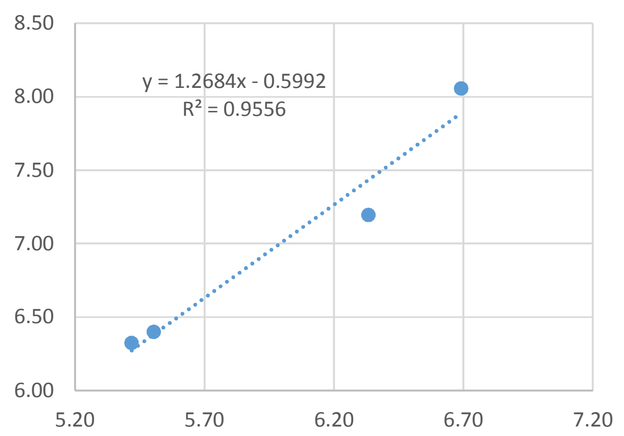
Another way to check unequal variances is to do a F statistic test where from the population, we find a ratio of the largest variance estimate to the smallest, s_{max}^2/s_{min}^2. The rule of thumb is that this ratio does not exceed three for equal variances. In this example, the F value = 3152.25/557.67 = 5.65 which was greater than 3. Hence, a transformation of data to stabilize the variances is necessary.

Using the treatment sample means and variances from Table 1, we have:

Treatment	1	2	3	4
Mean $\bar{y}_i$	562.5	804.75	225.5	245.75
$\ln(\bar{y}_i)$	6.332	6.691	5.418	5.504
Variance s_i^2	1333.67	3152.25	557.67	601.58
$\ln(s_i^2)$	7.196	8.056	6.324	6.400

A plot of $\ln(s_i^2)$ against $\ln(\bar{y}_i)$ as shown in Figure 2 below is nearly linear, so the slope of the curve will provide an estimate of q in equation (2).

Fig 2: A plot of $\ln(s_i^2)$ against $\ln(\text{mean } y_i)$



By taking q = 1.2684 from the gradient of the least square line, we have $(1-q/2)$ = 0.366. Hence, from the equation (2), a variance transformation is

$$h(y_{it}) = (y_{it})^{0.366}$$

Since $(y_{it})^{0.366}$ is close to $(y_{it})^{0.5}$ and since the square root of the data values is perhaps more meaningful than $(y_{it})^{0.366}$, we will try taking the square root transformation. The square roots of the data are shown in Table 4. We note that this transformation has stabilized the variances considerably, as evidenced by s_{max}^2/s_{min}^2 = 0.982/0.587 = 1.67 which is less than 3.

Table 4: Transformed data $\sqrt{y_{it}}$ for the battery experiment

Treatment	1	2	3	4
	24.536	29.377	15.232	15.330
	23.000	27.258	15.969	16.793
	23.108	27.803	14.142	15.427
	24.187	28.983	14.663	15.100
Mean	23.708	28.355	15.001	15.662
Std Deviation	0.769	0.991	0.783	0.766
Variance	0.592	0.982	0.614	0.587

Checks of the other model assumptions for the transformed data also reveal no severe problem and so we can now proceed the analysis using these transformed data.

It may be noted that the significance level and confidence levels will now be approximate, because the model has been changed based on the data. The transformed standard error SE = 0.745 instead of 33.60 before transformation.